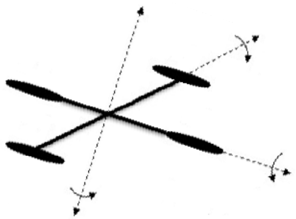
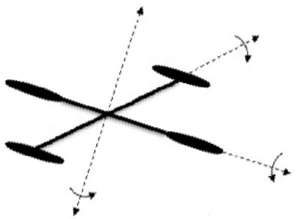
8장 Matrice -1



정의 8.1 행렬

행렬은 숫자나 함수의 직사각형 배열이다.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad (2)$$



- Matrices

열 (column)

행 (row)

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

크기 : 2*3

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

column vector

row vector

정방행렬
(square matrices)

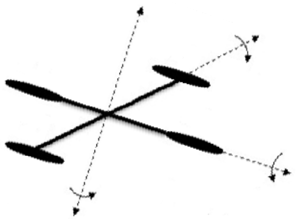
- Equality of Matrices

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal,** written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different.** Thus, matrices of different sizes are always different.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

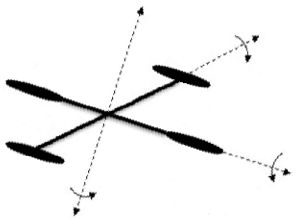


- Addition of Matrices

Addition of Matrices

The sum of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

$$\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$



- Scalar Multiplication of Matrices

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

$$\mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Rules for Matrix Addition and Scalar Multiplication

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \qquad \text{(written } \mathbf{A} + \mathbf{B} + \mathbf{C}\text{)}$$

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

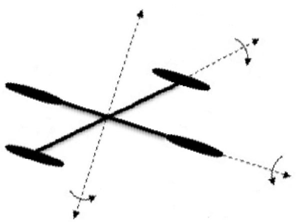
$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

$$(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$$

$$c(k\mathbf{A}) = (ck)\mathbf{A} \qquad \text{(written } ck\mathbf{A}\text{)}$$

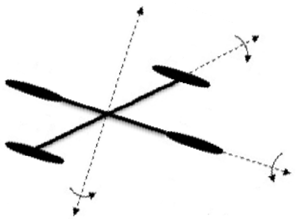
$$1\mathbf{A} = \mathbf{A}.$$



예제 1 상등성

(a) 행렬 $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$과 $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$는 같지 않다. 왜냐하면 첫 행렬의 크기는 2×2이고 두 번째 행렬의 크기는 2×3이기 때문이다.

(b) 행렬 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$와 $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$는 같지 않다. 왜냐하면 두 번째 행렬의 대응 원소들이 같지 않기 때문이다. ❑

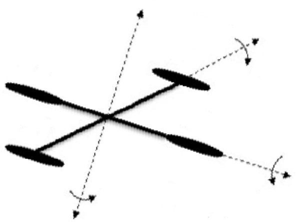


예제 2 두 행렬의 덧셈

(a) 행렬 $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix}$ 와 행렬 $\mathbf{B} = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ 의 합은 아래와 같다.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+4 & -1+7 & 3+(-8) \\ 0+9 & 4+3 & 6+5 \\ -6+1 & 10+(-1) & -5+2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{pmatrix}$$

(b) 행렬 $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ 와 $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ 의 합은 정의할 수 없다. 왜냐하면 행렬 $\mathbf{A}$ 의 크기와 행렬 $\mathbf{B}$ 의 크기가 다르기 때문이다. ❑



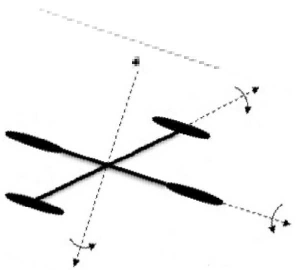
- Matrix Multiplication

$$\begin{array}{ccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times r] & = & [m \times r]. \end{array}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 3 + 8 \cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

$$[3 \quad 6 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19], \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} [3 \quad 6 \quad 1] = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$



CAUTION! Matrix Multiplication Is Not Commutative, AB $\neq$ BA in General

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}$$

(a) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$ *written* $k\mathbf{AB}$ *or* $\mathbf{A}k\mathbf{B}$

(b) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ *written* $\mathbf{ABC}$

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(d) $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

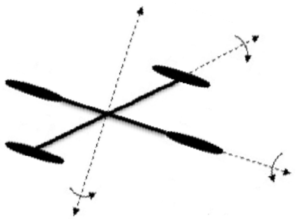
Computing Products Columnwise

$$AB = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

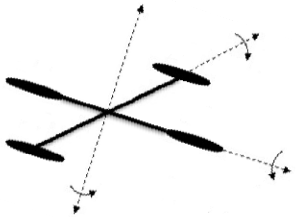


- Motivation of Multiplication by Linear Transformations

$$y_1 = a_{11}x_1 + a_{12}x_2$$

$$y_2 = a_{21}x_1 + a_{22}x_2$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$



예제 3 행렬의 곱셈

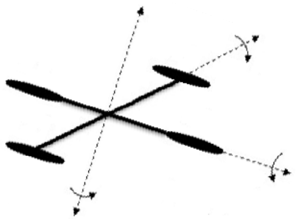
아래 두 개 행렬의 곱 **AB**를 구하라.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix}$

풀이 정의 8.6으로부터 다음을 얻는다.

(a) $$\mathbf{AB} = \begin{pmatrix} 4 \cdot 9 + 7 \cdot 6 & 4 \cdot (-2) + 7 \cdot 8 \\ 3 \cdot 9 + 5 \cdot 6 & 3 \cdot (-2) + 5 \cdot 8 \end{pmatrix} = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}$$

(b) $$\mathbf{AB} = \begin{pmatrix} 5 \cdot (-4) + 8 \cdot 2 & 5 \cdot (-3) + 8 \cdot 0 \\ 1 \cdot (-4) + 0 \cdot 2 & 1 \cdot (-3) + 0 \cdot 0 \\ 2 \cdot (-4) + 7 \cdot 2 & 2 \cdot (-3) + 7 \cdot 0 \end{pmatrix} = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$ ❑



- Transpose of Matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^\mathsf{T} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

$$[6 \quad 2 \quad 3]^\mathsf{T} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

(a) $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$

(c) $(c\mathbf{A})^\mathsf{T} = c\mathbf{A}^\mathsf{T}$

(d) $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}.$

- Symmetric and Skew-Symmetric Matrix

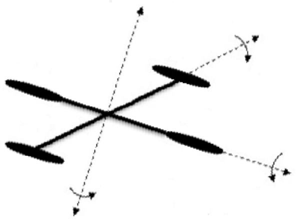
$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \text{ is symmetric, and } \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \text{ is skew-symmetric.}$$

- Triangular Matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$$

Upper triangular

Lower triangular

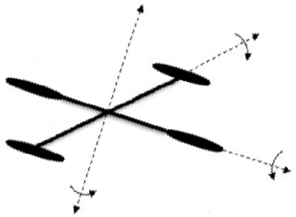


- Diagonal Matrices

Diagonal Matrix D. Scalar Matrix S. Unit Matrix I

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A}. \qquad \mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$



이 계는 동차이다 ↓

$$
\begin{aligned}
5x_1 - 9x_2 + x_3 &= 0 \\
x_1 + 3x_2 \qquad &= 0 \\
4x_1 + 6x_2 - x_3 &= 0
\end{aligned}
$$

이 계는 비동차이다 ↓

$$
\begin{aligned}
2x_1 + 5x_2 + 6x_3 &= 1 \\
4x_1 + 3x_2 - \ x_3 &= 9
\end{aligned}
$$

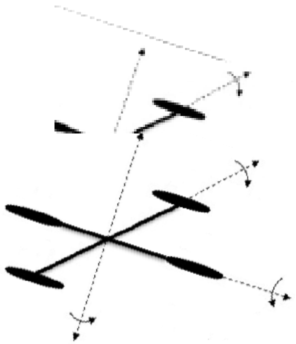
- Matrix Form of the Linear Systems, Coefficient matrix

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \rightarrow \quad \boxed{\mathbf{Ax} = \mathbf{b}}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

↓

Coefficient matrix

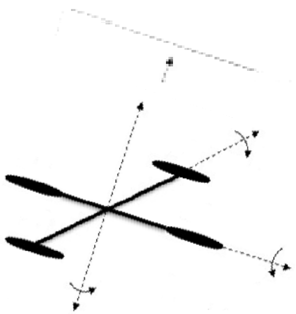


- Augmented Matrix

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

$$\widetilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$



예제 3 첨가행렬

(a) 첨가행렬 $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 2 \\ 4 & 7 & -1 & 8 \end{array}\right)$ 은 다음 선형계를 표현한다.

$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= 2 \\ 4x_1 + 7x_2 - \ \ x_3 &= 8 \end{aligned}$$

(b) 왼쪽의 선형계는 오른쪽의 선형계와 동일하다.

$$\begin{aligned} x_1 - 5x_3 &= -1 \\ 2x_1 + 8x_2 &= \ \ 7 \\ x_2 + 9x_3 &= \ \ 1 \end{aligned} \quad \text{은} \quad \begin{aligned} x_1 + 0x_2 - 5x_3 &= -1 \\ 2x_1 + 8x_2 + 0x_3 &= \ \ 7 \\ 0x_1 + \ \ x_2 + 9x_3 &= \ \ 1 \end{aligned} \quad \text{와 같다}$$

따라서 계의 행렬은 아래와 같다.

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -1 \\ 2 & 8 & 0 & 7 \\ 0 & 1 & 9 & 1 \end{array}\right)$$

❑

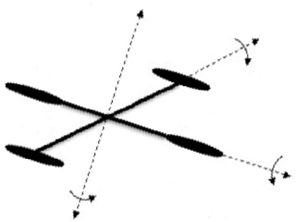
- 행 연산

행렬의 행 연산은 선형대수방정식의 해를 구할 때, 사용 함.

기본규칙

1. 어떤 두 행의 위치를 바꿀 수 있다.
2. 한 행에 0이 아닌 상수를 곱할 수 있다.
3. 어떤 두 행을 서로 더할 수 있다.

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & -3 \\ 1 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -2 \\ 1 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -2 \\ -2 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -2 \\ 0 & 3 & -7 \end{pmatrix}$$

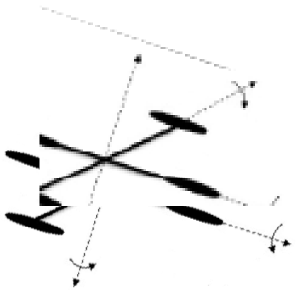


- Gauss Elimination and Back Substitution

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30. \end{aligned} \quad \xrightarrow{\text{Augmented}} \quad \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \rightarrow *\frac{1}{2} \quad \Rightarrow \quad \begin{bmatrix} 1 & \frac{5}{2} & 1 \\ -4 & 3 & -30 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 + \frac{5}{2}x_2 &= 1 \\ -4x_1 + 3x_2 &= -30 \end{aligned}$$

$$\begin{bmatrix} 1 & \frac{5}{2} & 1 \\ -4 & 3 & -30 \end{bmatrix} \rightarrow *\frac{1}{4} \quad \Rightarrow \quad \begin{bmatrix} 1 & \frac{5}{2} & 1 \\ -1 & \frac{3}{4} & -\frac{15}{2} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 + \frac{5}{2}x_2 &= 1 \\ -x_1 + \frac{3}{4}x_2 &= -\frac{15}{2} \end{aligned}$$



$$+ \begin{bmatrix} 1 & \frac{5}{2} & 1 \\ -1 & \frac{3}{4} & -\frac{15}{2} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & \frac{5}{2} & 1 \\ 0 & \frac{13}{4} & -\frac{13}{2} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 + \frac{5}{2}x_2 &= 1 \\ \frac{13}{4}x_2 &= -\frac{13}{2} \end{aligned}$$

$$\begin{bmatrix} 1 & \frac{5}{2} & 1 \\ 0 & \frac{13}{4} & -\frac{13}{2} \end{bmatrix} \rightarrow * \frac{4}{13} \Rightarrow \begin{bmatrix} 1 & \frac{5}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + \frac{5}{2}x_2 &= 1 \\ x_2 &= -2 \end{aligned}$$

$$- \begin{bmatrix} 1 & \frac{5}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow * \frac{5}{2} \Rightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 6 \\ x_2 &= -2 \end{aligned}$$

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30. \end{aligned}$$

예제 4 사다리꼴 행렬

(a) 아래의 첨가행렬은 행사다리꼴 형태이다.

$$\left(\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ 그리고 } \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

독자는 이 형태에 대해 3가지 기준이 충족되는 것을 확인할 수 있어야 한다.

(b) 아래의 첨가행렬은 기약 행사다리꼴 형태이다.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ 그리고 } \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

선도 요소 1을 가진 열에서 나머지 요소들은 모두 영인 것에 주의하라. ❑

기호	의미
R_{ij}	행 i와 j를 교환한다.
cR_i	i번째 행에 영이 아닌 상수 c를 곱한다.
cR_i+R_j	i번째 행에 영이 아닌 상수 c를 곱하고 j번째 행에 더한다.

$$\begin{aligned} 2x_1 + 6x_2 + \ \ x_3 &= 7 \\ x_1 + 2x_2 - \ \ x_3 &= -1 \\ 5x_1 + 7x_2 - 4x_3 &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right) \overset{R_{12}}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right)$$

$$\overset{\substack{-2R_1+R_2 \\ -5R_1+R_3}}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right) \overset{\frac{1}{2}R_2}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & -3 & 1 & 14 \end{array}\right)$$

$$\overset{3R_2+R_3}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & \frac{11}{2} & \frac{55}{2} \end{array}\right) \overset{\frac{2}{11}R_3}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$$\begin{aligned} x_1 + 2x_2 - \ \ x_3 &= -1 \\ x_2 + \frac{3}{2}x_3 &= \frac{9}{2} \\ x_3 &= 5 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right) \overset{-2R_2+R_1}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right) \overset{\substack{-4R_3+R_1 \\ -\frac{3}{2}R_3+R_2'}}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$$x_1 = 10,\ x_2 = -3,\ x_3 = 5$$

예제 6 Gauss-Jordan 소거법

Gauss-Jordan 소거법을 사용하여 아래 방정식을 풀라.

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 &= -7\\
4x_1 + x_2 + 3x_3 &= 5\\
2x_1 - 5x_2 + 7x_3 &= 19
\end{aligned}
$$

풀이

$$
\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7\\ 4 & 1 & 3 & 5\\ 2 & -5 & 7 & 19 \end{array}\right)
\overset{\substack{-4R_1+R_2\\ -2R_1+R_3}}{\Longrightarrow}
\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7\\ 0 & -11 & 11 & 33\\ 0 & -11 & 11 & 33 \end{array}\right)
$$

$$
\overset{\substack{-\frac{1}{11}R_2\\ -\frac{1}{11}R_3}}{\Longrightarrow}
\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7\\ 0 & 1 & -1 & -3\\ 0 & 1 & -1 & -3 \end{array}\right)
\overset{\substack{-3R_2+R_1\\ -R_2+R_3}}{\Longrightarrow}
\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2\\ 0 & 1 & -1 & -3\\ 0 & 0 & 0 & 0 \end{array}\right)
$$

$x_1 = 2 - t,\ x_2 = -3 + t,\ x_3 = t.$

예제 7 **해가 없는 계**

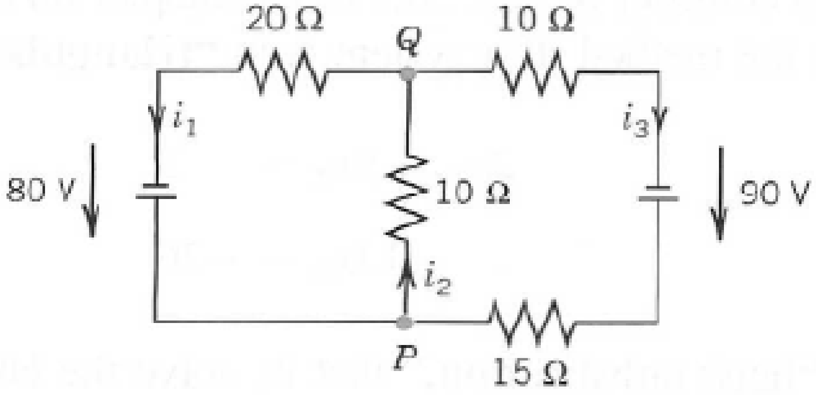
아래 방정식을 풀라.

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
4x_1 - x_2 &= -6 \\
2x_1 - 3x_2 &= 8
\end{aligned}
$$

풀이 계의 행렬에 Gauss-Jordan 소거법을 적용하면, 아래와 같이 종료된다.

$$
\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right) \overset{\text{행 연산}}{\Rightarrow} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right)
$$

마지막 행렬의 세 번째 행은 $0x_1+0x_2=16$(또는 $0=16$)이다. x_1과 x_2에 어떤 수를 넣어도 만족시킬 수 없으므로, 계는 해가 없다고 결론짓는다. ❑



$$
\begin{aligned}
\text{Node } P: &\quad i_1 - i_2 + i_3 = 0\\
\text{Node } Q: &\quad -i_1 + i_2 - i_3 = 0\\
\text{Right loop:} &\quad 10i_2 + 25i_3 = 90\\
\text{Left loop:} &\quad 20i_1 + 10i_2 = 80
\end{aligned}
$$

Augmented Matrix $\bar{\mathbf{A}}$

Pivot 1 → (row 1); Eliminate → (rows 2–4)

$$
\left[\begin{array}{ccc|c}
\textcircled{1} & -1 & 1 & 0\\
-1 & 1 & -1 & 0\\
0 & 10 & 25 & 90\\
20 & 10 & 0 & 80
\end{array}\right]
$$

Equations

Pivot 1 → (first equation); Eliminate → (remaining equations)

$$
\begin{aligned}
x_1 - x_2 + x_3 &= 0\\
-x_1 + x_2 - x_3 &= 0\\
10x_2 + 25x_3 &= 90\\
20x_1 + 10x_2 &= 80.
\end{aligned}
$$

Augmented Matrix $\tilde{\mathbf{A}}$

Pivot 1 →, Eliminate →

$$\begin{bmatrix} \textcircled{1} & -1 & 1 & \vert & 0 \\ -1 & 1 & -1 & \vert & 0 \\ 0 & 10 & 25 & \vert & 90 \\ 20 & 10 & 0 & \vert & 80 \end{bmatrix}$$

Equations

Pivot 1 →, Eliminate →

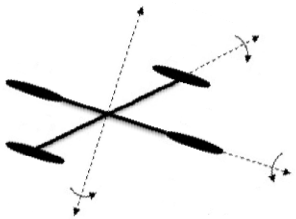
$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 1 & \vert & 0 \\ 0 & 0 & 0 & \vert & 0 \\ 0 & 10 & 25 & \vert & 90 \\ 0 & 30 & -20 & \vert & 80 \end{bmatrix}$$

Row 2 + Row 1

Row 4 − 20 Row 1

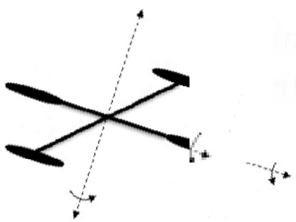
$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$



$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \quad \begin{array}{l} \\ \text{Row 2 + Row 1} \\ \\ \text{Row 4} - 20 \text{ Row 1} \end{array} \quad \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & \textcircled{10} & 25 & 90 \\ 0 & \boxed{30} & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate } 30x_2 \longrightarrow \\ \\ \end{array} \quad \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ \textcircled{10x_2} + 25x_3 &= 90 \\ \boxed{30x_2} - 20x_3 &= 80 \\ 0 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \text{Row 3} - 3 \text{ Row 2} \\ \\ \end{array} \quad \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0 \end{aligned}$$



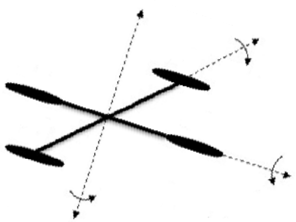
$$\begin{aligned} -95x_3 &= -190 \\ 10x_2 + 25x_3 &= 90 \\ x_1 - x_2 + x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_3 &= i_3 = \boxed{2 \text{ [A]}} \\ x_2 &= \tfrac{1}{10}(90 - 25x_3) = i_2 = \boxed{4 \text{ [A]}} \\ x_1 &= x_2 - x_3 = i_1 = \boxed{2 \text{ [A]}} \end{aligned}$$

$$\begin{pmatrix} -3 & 2 & 2 \\ 1 & 4 & -6 \\ 0 & -2 & 2 \end{pmatrix} X = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} -3 & 2 & 2 & 8 \\ 1 & 4 & -6 & 1 \\ 0 & -2 & 2 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ -3 & 2 & 2 & 8 \\ 0 & -2 & 2 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & 14 & -16 & 11 \\ 0 & -2 & 2 & -2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & 1 & -\frac{8}{7} & \frac{11}{14} \\ 0 & -2 & 2 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{10}{7} & -\frac{15}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{11}{14} \\ 0 & 0 & -\frac{2}{7} & -\frac{3}{7} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{10}{7} & -\frac{15}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{11}{14} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$



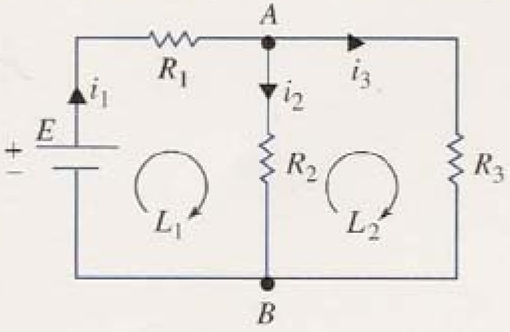


그림 8.3 전기회로망

$$i_1 - i_2 - i_3 = 0 \qquad\qquad i_1 - i_2 - i_3 = 0$$

$$E - i_1R_1 - i_2R_2 = 0 \quad \text{또는} \quad i_1R_1 + i_2R_2 = E$$

$$i_2R_2 - i_3R_3 = 0 \qquad\qquad i_2R_2 - i_3R_3 = 0$$

예제 8 회로망의 전류

저항값(ohm)이 $R_1=10$, $R_2=20$, $R_3=10$이고 기전력은 $E=12$ V일 때 (3)의 계를 Gauss-Jordan 소거법을 사용하여 풀라.

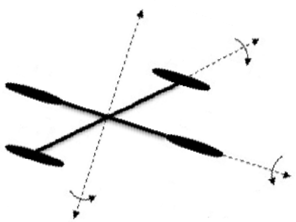
풀이 풀고자 하는 계는 아래와 같다.

$$\begin{aligned} i_1 - \quad i_2 - \quad i_3 &= 0 \\ 10i_1 + 20i_2 \qquad &= 12 \\ 20i_2 - 10i_3 &= 0 \end{aligned}$$

이 경우, Gauss-Jordan 소거법에 따르면 다음과 같은 결과가 나온다.

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 10 & 20 & 0 & 12 \\ 0 & 20 & -10 & 0 \end{array}\right) \overset{\text{행 연산}}{\Rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{18}{25} \\ 0 & 1 & 0 & \frac{6}{25} \\ 0 & 0 & 1 & \frac{12}{25} \end{array}\right)$$

여기에서 3개 가지에 흐르는 전류(암페어, A)는 $i_1=\frac{18}{25}=0.72$, $i_2=\frac{6}{25}=0.24$, $i_3=\frac{12}{25}=0.48$이다. ❑



정의 8.8 **행렬의 계수**

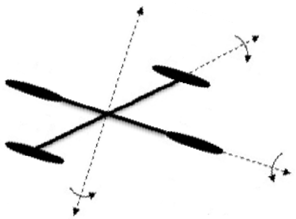
$m \times n$ 인 행렬 $\mathbf{A}$ 의 계수(rank)는 $\mathbf{A}$ 안에 있는 일차독립인 행벡터들의 최대 개수이고, rank($\mathbf{A}$)라고 표기한다.

예제 1 3×4 행렬의 계수

3×4 행렬 $\mathbf{A}$ 를 고려한다.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad (1)$$

행벡터 $\mathbf{u}_1=(-1 \;\; 1 \;\; -1 \;\; 3)$, $\mathbf{u}_2=(2 \;\; -2 \;\; 6 \;\; 8)$, $\mathbf{u}_3=(3 \;\; 5 \;\; -7 \;\; 8)$에서 $4\mathbf{u}_1-\frac{1}{2}\mathbf{u}_2+\mathbf{u}_3=\mathbf{0}$ 가 되고, 정의 7.7 의 관점에서 보면 $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ 는 일차종속인 것으로 결론 내릴 수 있다. 반면에 행벡터의 집합 $\mathbf{u}_1$ 과 $\mathbf{u}_2$ 는 일차독립이다. 따라서 정의 8.8 에 따라 rank($\mathbf{A}$)=2 이다. ❑



예제 2 행축소에 의한 계수—예제 1 의 재현

Gauss 소거법을 사용하여 계의 해를 구할 때 선형방정식들의 계로 된 첨가행렬을 행축소하여 사다리꼴로 만드는 것과 정확하게 같은 방법으로 **A** 의 행을 축소하여 행사다리꼴 행렬 **B** 로 만든다. 예제 1 의 행렬(1)을 사용하여 기본 행 연산을 수행하면 아래와 같다.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \xRightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 2 & -4 & -1 \end{pmatrix} \xRightarrow[-\frac{1}{4}R_2]{\frac{1}{2}R_2+R_3} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

마지막 행렬은 행사다리꼴이고 영이 아닌 행을 2개 가지고 있으므로, 정리 8.4의 (*iii*)에 따라서 **A** 의 계수는 2이다. ❑

예제 3 일차독립/종속

3차원 공간 R^3의 벡터 $\mathbf{u}_1 = \langle 2, 1, 1 \rangle$, $\mathbf{u}_2 = \langle 0, 3, 0 \rangle$, $u_3 = \langle 3, 1, 2 \rangle$의 집합이 일차독립 또는 일차종속인지를 판정하라.

풀이 주어진 벡터를 행으로하여 행렬 **A**를 구성하고, **A**를 계수 3인 행사다리꼴 **B**로 행을 축소한다면, 벡터의 집합은 일차독립인 것을 위의 논의로부터 확신할 수 있다. 만약 rank(**A**) < 3이면, 벡터의 집합은 일차종속이다. 이 경우, 쉽게 기약 행사다리꼴로 행축소를 할 수 있다.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{\text{행 연산}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

그래서 rank(**A**)=3이고, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$는 일차독립이다. ❑

- Linear Independence and Dependence of Vectors

$\vec{a} = k\vec{b}$ → Linear Dependence

$\vec{a} \neq k\vec{b}$ → Linear Dependence

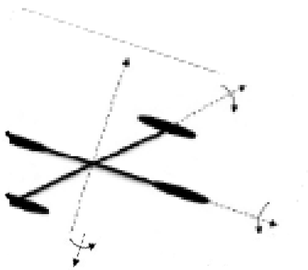
$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}.$$

$c_i \neq 0$ → Linear Dependence

$c_i = 0$ → Linear Independence

$$\mathbf{a}_{(1)} = [\;3 \quad 0 \quad 2 \quad 2]$$
$$\mathbf{a}_{(2)} = [\;-6 \quad 42 \quad 24 \quad 54]$$
$$\mathbf{a}_{(3)} = [\;21 \quad -21 \quad 0 \quad -15]$$

Linear Dependence



$$6\mathbf{a}_{(1)} - \tfrac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0}.$$

- Rank of a Matrix

The **rank** of a matrix **A** is the maximum number of linearly independent row vectors of **A**. It is denoted by rank **A**.

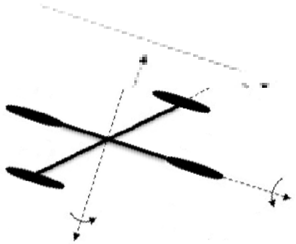
$$\mathbf{a}_{(1)} = [\ 3 \quad 0 \quad 2 \quad 2]$$
$$\mathbf{a}_{(2)} = [-6 \quad 42 \quad 24 \quad 54]$$
$$\mathbf{a}_{(3)} = [\ 21 \quad -21 \quad 0 \quad -15]$$

Linear Independence or Dependence ?

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Rank of A : 2

IF Linear Independence → Rank of A : 3



Determination of Rank

For the matrix in Example 2 we obtain successively

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} \\ \text{Row 2} + 2 \text{ Row 1} \\ \text{Row 3} - 7 \text{ Row 1} \end{matrix}$$

$$\begin{bmatrix} \textcircled{3} & 0 & 2 & 2 \\ 0 & \textcircled{42} & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ \\ \text{Row 3} + \frac{1}{2} \text{ Row 2} \end{matrix}$$

Since rank is defined in terms of two vectors, we immediately have the useful